

MOTT LAW AS UPPER BOUND FOR A RANDOM WALK IN A RANDOM ENVIRONMENT.

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ABSTRACT. We consider a random walk on the support of an ergodic simple point process on \mathbb{R}^d , $d \geq 2$, furnished with independent energy marks. The jump rates of the random walk decay exponentially in the jump length and depend on the energy marks via a Boltzmann-type factor. This is an effective model for the phonon-induced hopping of electrons in disordered solids in the regime of strong Anderson localization. Under some technical assumption on the point process we prove an upper bound for the diffusion matrix of the random walk in agreement with Mott law. A lower bound for $d \geq 2$ in agreement with Mott law was proved in [8].

Key words: disordered system, Mott law, random walk in random environment, marked point process, stochastic domination, continuum percolation.

1. INTRODUCTION

1.1. Physical motivations. Phonon-assisted electron transport in disordered solids in which the Fermi level (set equal to 0 below) lies in a region of strong Anderson localization can be modeled by Mott variable-range hopping of the following interacting particle system in a random environment [17]. The environment is given by $\xi := (\{x_i\}, \{E_i\})$, where $\{x_i\}$ is an infinite and locally finite set of points in \mathbb{R}^d such that each point x_i is labeled by an energy mark E_i belonging to some finite interval. Given ξ , particles can lie only at points of $\{x_i\}$ and perform random walks on $\{x_i\}$ with hard-core interaction. The probability rate for a jump of a particle at x to the vacant site y , with $x \neq y$ in $\{x_i\}$, is given by

$$r_{x,y}(\xi) = r_0 \exp \{-2|x - y|/\ell_* - \beta\{E_y - E_x\}_+\}, \quad (1.1)$$

where E_z is defined by E_i for $z = x_i$ and $\{E_y - E_x\}_+ = \max\{E_y - E_x, 0\}$. Above $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d , $\beta = 1/(kT)$, ℓ_* denotes the localization radius of wavefunctions and r_0 is a constant depending on the material but which depends only weakly on β , $|x - y|$ and the energies E_x, E_y . Without loss of generality, in what follows we set $\ell_* = 2$, $r_0 = 1$.

The disorder of the solid is modeled by the randomness of the environment. The points $\{x_i\}$ correspond to the *impurities* of the disordered solid and the electron Hamiltonian has exponentially localized quantum eigenstates with localization centers x_i if the corresponding energies E_i are close to the Fermi level. The random set $\{x_i\}$ is a stationary simple point process with finite intensity (i.e. with finite mean number of points in finite boxes) and its stationarity reflects the homogeneity of the medium. Conditioned to $\{x_i\}$, the energy marks $\{E_i\}$ are supposed to be i.i.d. random variables taking value in a finite interval (set in what follows equal to $[-1, 1]$) with common distribution ν , such that $\nu(dE) \sim c|E|^\alpha dE$, $|E| \ll 1$, for a suitable nonnegative exponent α . The independence of the energy marks is compatible with Poisson level statistics, which is a general rough indicator for the localization regime and has been proven to hold for an Anderson model [13]. The exponent α allows to model a possible Coulomb pseudogap in the density of

states [17]. In particular, the physically relevant possible values of α are 0 and $d - 1$, the latter corresponding to the Coulomb pseudogap.

The DC conductivity matrix $\sigma(\beta)$, measuring the linear response of the solid to a uniform external electric field, would vanish if it were not for the lattice vibrations (phonons) at nonzero temperature. For an isotropic solid at low temperature and with dimension $d \geq 2$, Mott law [14], [15] predicts that the conductivity decays exponentially as

$$\sigma(\beta) \sim \exp \left\{ -c \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \mathbb{I}, \quad \beta \gg 1, \quad (1.2)$$

where c is a β -independent positive constant and \mathbb{I} denotes the identity matrix. The original derivation of (1.2) is based on an heuristic optimization argument, alternative and more robust derivations have been proposed after that (see [1], [17] and references therein). Finally, we point out that due to the Einstein relation between $\sigma(\beta)$ and the bulk diffusion matrix $D_{\text{bulk}}(\beta)$ [18], (1.2) is equivalent to the asymptotic behavior

$$D_{\text{bulk}}(\beta) \sim \exp \left\{ -c \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \mathbb{I}, \quad \beta \gg 1. \quad (1.3)$$

A mean field version of Mott variable-range hopping at small temperature is given by the continuous-time random walk X_t^ξ on $\{x_i\}$ such that the jump from x to y , $x \neq y$ in $\{x_i\}$, has probability rate given by

$$c_{x,y}(\xi) = \exp \{ -|x - y| - \beta(|E_x - E_y| + |E_x| + |E_y|)/2 \}. \quad (1.4)$$

We will call X_t^ξ *Mott variable-range random walk*.

The choice of the transition rates $c_{x,y}(\xi)$ comes from the fact that, at small temperature,

$$c_{x,y}(\xi) = r_{x,y}(\xi) \mu(\eta_x = 1) \mu(\eta_y = 0) (1 + o(1)), \quad \beta \gg 1, \quad (1.5)$$

where η_x is the particle number at site x in the above particle system, while μ is the Gibbs measure of the particle system w.r.t. the Hamiltonian $H = \sum_i E_i \eta_{x_i}$ with zero Fermi level, i.e. μ is the product measure on $\{0, 1\}^{\{x_i\}}$ such that

$$\mu(\eta_{x_i} = 1) = e^{-\beta E_i} / (1 + e^{-\beta E_i}).$$

In (1.5), the error term is negligible as $\beta \rightarrow \infty$ uniformly in x, y belonging to a finite volume (as in realistic solids), for a typical environment ξ . Another derivation of the mean field version (1.4) from the original Mott variable range hopping (1.1) has been obtained in [12] (see also [1][Section IV]) by reduction to a random resistor network.

The analogous of Mott law (1.3) for Mott variable-range random walk is given by

$$D(\beta) \sim \exp \left\{ -c \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \mathbb{I}, \quad \beta \gg 1, \quad (1.6)$$

where $D(\beta)$ denotes the diffusion matrix of the random walk. For dimension $d \geq 2$, a lower bound of $D(\beta)$ in agreement with (1.6) has been recently proven in [8]. The present work addresses to the problem of rigorously deriving an upper bound of $D(\beta)$ in agreement with (1.6). The one dimensional case present special features and rigorous results on the corresponding Mott law have been obtained in [5].

1.2. Model and results. Let us give a precise definition of Mott variable-range random walk X_t^ξ , generalizing the choice of jump rates (1.4). The environment $\xi = (\{x_i\}, \{E_i\})$ is defined as follows. Let $\{x_i\}$ be a simple point process, i.e. a random locally finite subset of \mathbb{R}^d , and its law is the Palm distribution $\hat{\mathcal{P}}_0$ associated to the law $\hat{\mathcal{P}}$ of a stationary simple point process on \mathbb{R}^d with finite intensity. Given $\{x_i\}$, the energy marks $\{E_i\}$ are i.i.d. random variables having value in $[-1, 1]$ and common law ν (restrictions on ν will

be specified later). The law \mathcal{P}_0 of the environment is the so called ν -randomization of the Palm distribution $\hat{\mathcal{P}}_0$ associated to $\hat{\mathcal{P}}$.

The reason for considering Palm distributions is the following: in order to make the random walk start at a fixed point, taken equal to the origin below, we condition to contain the origin the stationary point process of impurities introduced in the previous subsection. As discussed in Section 2, if the stationary process has finite intensity and law $\hat{\mathcal{P}}$, the law of the resulting process is given by the Palm distribution $\hat{\mathcal{P}}_0$ associated to $\hat{\mathcal{P}}$.

In what follows, we write ξ for a generic locally finite subset of $\mathbb{R}^d \times [-1, 1]$ such that ξ has at most one point in each fiber $\{x\} \times [-1, 1]$, and we write $\hat{\xi}$ for a generic locally finite subset of \mathbb{R}^d . Both ξ and $\hat{\xi}$ can be identified with the counting measures $\sum_{(x,E) \in \xi} \delta_{(x,E)}$ and $\sum_{x \in \hat{\xi}} \delta_x$ respectively. Moreover, when there is no ambiguity, given ξ its spatial projection on \mathbb{R}^d will be denoted by $\hat{\xi}$. We finally recall that the κ -moment ρ_κ of $\hat{\mathcal{P}}$, $\kappa > 0$, is defined as

$$\rho_\kappa := \mathbf{E}_{\hat{\mathcal{P}}} \left(|\hat{\xi} \cap [0, 1]^d|^\kappa \right).$$

Then $\rho := \rho_1$ is called the intensity of the process $\hat{\mathcal{P}}$.

Given a realization of the environment ξ , Mott variable-range random walk X_t^ξ is defined as the continuous-time random walk on $\hat{\xi} = \{x_i\}$ starting at the origin and jumping from x to y , $x \neq y$ in $\{x_i\}$, with probability rate

$$c_{x,y}(\xi) = \exp \{ -|x - y| - \beta u(E_x, E_y) \}, \quad (1.7)$$

where the function u satisfies

$$\kappa_1(|E_x| + |E_y|) \leq u(E_x, E_y) \leq \kappa_2(|E_x| + |E_y|) \quad (1.8)$$

for some positive constants $\kappa_1 \leq \kappa_2$. To simplify the notation, it is convenient to set $c_{x,x}(\xi) \equiv 0$ for all $x \in \{x_i\}$.

The law \mathbb{P}^ξ of X_t^ξ is characterized by the following identities:

$$\begin{aligned} \mathbb{P}^\xi(X_0^\xi = 0) &= 1, \\ \mathbb{P}^\xi(X_{t+dt}^\xi = y \mid X_t^\xi = x) &= c_{x,y}(\xi)dt + o(dt), \quad t \geq 0, \quad x \neq y, \\ \mathbb{P}^\xi(X_{t+dt}^\xi = x \mid X_t^\xi = x) &= 1 - \sum_z c_{x,z}(\xi)dt + o(dt), \quad t \geq 0. \end{aligned}$$

Equivalently, the dynamics of X_t^ξ can be described as follows: after arriving at site x the particle waits an exponential time with parameter

$$\lambda_x(\xi) = \sum_z c_{x,z}(\xi), \quad (1.9)$$

and then jumps to site y , $y \neq x$, with probability

$$\frac{c_{x,y}(\xi)}{\lambda_x(\xi)}. \quad (1.10)$$

By standard methods (see e.g. [3], [8][Appendix A]), one can check that the random walk X_t^ξ is well-defined for \mathcal{P}_0 -a.a. ξ as soon as $\hat{\mathcal{P}}$ is ergodic w.r.t. spatial translations and $\mathbf{E}_{\mathcal{P}_0}(\lambda_0(\xi)) < \infty$. As proven in [8] (see also Lemma 1 below) this last condition is equivalent to require that $\rho_2 < \infty$.

Assuming $\hat{\mathcal{P}}$ to be ergodic, ρ_{12} to be finite and under some additional technical assumption on the law of the environment \mathcal{P}_0 , in [8] the authors prove that the diffusively rescaled

random walk X^ξ converges in \mathcal{P}_0 -probability to a Brownian motion whose covariance matrix coincides with $D(\beta)$, where $D(\beta)$ is the diffusion matrix of X^ξ defined as the unique symmetric matrix such that

$$(a \cdot D(\beta)a) = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbf{E}_{\mathcal{P}_0} \left(\mathbf{E}_{\mathbf{P}_0^\xi} \left((X_t^\xi \cdot a)^2 \right) \right), \quad a \in \mathbb{R}^d. \quad (1.11)$$

Moreover, they prove the following variational characterization of $D(\beta)$:

$$(a, D(\beta)a) = \inf_{f \in L^\infty(\mathcal{P}_0)} \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) (a \cdot x - \nabla_x f(\xi))^2, \quad a \in \mathbb{R}^d, \quad (1.12)$$

where the set $\hat{\xi}$ is defined as the spatial projection of ξ , i.e. $\hat{\xi} = \{x_i\}$, and is identified with the counting measure $\sum_i \delta_{x_i}$. Moreover, the gradient $\nabla_x f(\xi)$ is defined as

$$\nabla_x f(\xi) = f(S_x \xi) - f(\xi), \quad S_x \xi := \{(x_i - x, E_i)\}.$$

In addition, for $d \geq 2$ and assuming that

$$\nu([-E, E]) \geq c_0 |E|^{\alpha+1}, \quad \forall E \in [-1, 1], \quad (1.13)$$

for some positive constant c_0 and some exponent $\alpha \geq 0$, the authors prove a lower bound on $D(\beta)$: for β large enough it holds

$$D(\beta) \geq c_1 \beta^{-c_2} \exp \left\{ -C \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} \mathbb{I}, \quad (1.14)$$

where c_1, c_2, C are positive constants independent of β . The above bound is in agreement with Mott law (1.6).

We note that the requirement $\alpha > 0$ in [8][Theorem 1] is due to a typing error. Moreover it is simple to check that, although in [8] the transition rates $c_{x,y}(\xi)$ are defined as in (1.4), all the results in [8] remain true for transition rates $c_{x,y}(\xi)$ defined as in (1.7) and assuming only that $\alpha > -1$.

Our main result consists in an upper bound for $D(\beta)$ in agreement with Mott law. Roughly, we claim that if (1.13) holds with inverted sign, then also (1.14) remains valid with inverted sign. In order to precisely state our technical assumptions we fix some notation.

Given $p \in [0, 1]$ and a simple point process with law $\hat{\mathcal{P}}$, its p -*thinning* is the simple point process obtained as follows: for each realization $\hat{\xi}$ of the process with law $\hat{\mathcal{P}}$ erase each point independently with probability $1 - p$. We will write $\hat{\mathcal{P}}^{(p)}$ for the law of the p -thinning of $\hat{\mathcal{P}}$.

Finally, we recall that the *Poisson point process* on \mathbb{R}^d with intensity $\rho > 0$ is a random locally finite subset $\hat{\xi} \subset \mathbb{R}^d$ such that (i) for any $A \subset \mathbb{R}^d$ Borel and bounded, the cardinality $\hat{\xi}(A)$ is a Poisson random variable with expectation $\rho \ell(A)$ where $\ell(A)$ is the Lebesgue measure of A ; (ii) for any disjoint Borel subsets $A_1, \dots, A_n \subset \mathbb{R}^d$, $\hat{\xi}(A_1), \dots, \hat{\xi}(A_n)$ are independent random variables. We denote by $\hat{\mathcal{P}}_\rho$ the law of the Poisson point process with intensity ρ . The process $\hat{\mathcal{P}}_\rho$ is stationary.

We can finally state our main result:

Theorem 1. *Let \mathcal{P}_0 be the ν -randomization of the Palm distribution $\hat{\mathcal{P}}_0$ associated to a stationary simple point process on \mathbb{R}^d , $d \geq 2$, with law $\hat{\mathcal{P}}$ and finite intensity ρ , and let the following conditions be satisfied:*

- (i) For some constants $c_0 > 0$ and $\alpha > -1$

$$0 < \nu([-E, E]) \leq c_0 |E|^{\alpha+1}, \quad \forall E \in (0, 1]; \quad (1.15)$$

- (ii) There exist positive constants ρ', K and there exists $p \in (0, 1]$ such that, setting

$$\Lambda_K(x) = x + [-K/2, K/2]^d$$

and defining the random field Y as

$$Y = \{Y(x) : x \in K\mathbb{Z}^d\}, \quad Y(x) := \hat{\xi}(\Lambda_K(x)), \quad (1.16)$$

then the law of Y when $\hat{\xi}$ is chosen with law $\hat{\mathcal{P}}^{(p)}$ (the p -thinning of $\hat{\mathcal{P}}$) is stochastically dominated by the law of Y when $\hat{\xi}$ is chosen with law $\hat{\mathcal{P}}_{\rho'}$ (the Poisson point process with density ρ').

Then the $d \times d$ symmetric matrix $D(\beta)$ solving the variational problem (1.12) admits the following upper bound for β large enough:

$$D(\beta) \leq c_1 \beta^{c_2} \exp\left(-C \beta^{\frac{\alpha+1}{\alpha+1+d}}\right) \mathbb{I}, \quad (1.17)$$

for suitable β -independent positive constants c_1, c_2, C .

We recall that due to Strassen theorem the stochastic domination assumption in condition (ii) above is equivalent to the fact that one can construct on the same probability space processes $Y_1 = \{Y_1(x) : x \in K\mathbb{Z}^d\}$ and $Y_2 = \{Y_2(x) : x \in K\mathbb{Z}^d\}$ in such a way that

$$Y_1(x) \leq Y_2(x), \quad \forall x \in K\mathbb{Z}^d, \quad (1.18)$$

the law of Y_1 equals the law of Y when $\hat{\xi}$ is chosen with law $\hat{\mathcal{P}}^{(p)}$ and the law of Y_2 equals the law of Y when $\hat{\xi}$ is chosen with law $\hat{\mathcal{P}}_{\rho'}$.

Theorem 1 applies to the case that $\hat{\mathcal{P}}$ is or is dominated by a stationary Poisson point process $\hat{\mathcal{P}}_{\rho'}$, i.e. when one can define random sets (ξ, ξ') such that $\xi \subset \xi'$ almost surely and ξ, ξ' have marginal distributions $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_{\rho'}$ respectively. An example is given by Gibbsian random point fields with repulsive interactions (cf. [9] and [4][Section 5]).

Moreover, the above theorem covers also the case of thinnings of point processes with uniform bounds on the local density, as for example diluted crystals. Note that in this case the point process $\hat{\mathcal{P}}$ is not stochastically dominated by any stationary Poisson point process. We refer to Section 6 for a more detailed discussion.

1.3. Overview. In Section 2 we recall some definitions and results about point processes (see [6], [8] for more details) and state some technical results needed later on.

The proof of Theorem 1 is based on the variational formula (1.12) since for each fixed function $f \in L^\infty(\mathcal{P}_0)$ the r.h.s. in (1.12) gives an upper bound on $(a, D(\beta)a)$. In Section 3 we derive an upper bound on $D(\beta)$ by taking special test functions f in the r.h.s. of (1.12). The choice of such test functions is inspired by [16] and is related to the percolation approach of [1], [8]. In Section 4 we first show that the above upper bound together with some scaling arguments leads to (1.17) if \mathcal{P}_0 is the ν -randomization of the Palm distribution associated to a stationary Poisson point process. In Section 5 we extend the proof to general point processes satisfying assumption (ii).

As conjectured in the introduction of [8], the leading contribution to the conductivity of the medium as $\beta \uparrow \infty$ comes from a subset of impurities that converges to a Poisson point process under suitable space rescaling. Due to the relevance of Poisson point processes in

Mott's law and since in the Poissonian case the proof of Theorem 1 is more transparent and simple, we have preferred to treat the Poissonian case and the general case separately.

Finally, in Section 6 we show that Theorem 1 can be applied to thinnings of point processes with uniform bounds on the local density as diluted crystals.

For a more detailed discussion about mathematical aspects and physical motivations of Mott random walk we refer to [8], [17] and references therein.

2. SIMPLE POINT PROCESSES

In this section we recall some basic definitions and results about simple point processes, referring to [6], [8] for more details.

In what follows, given a topological set Y we write $\mathcal{B}(Y)$ for the σ -algebra of its Borel subsets. We denote $\hat{\mathcal{N}}$ the space of simple counting measures $\hat{\xi}$ on \mathbb{R}^d , i.e. integer-valued measures such that $\hat{\xi}(B) < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$, and $\hat{\xi}(x) \in \{0, 1\}$ for all $x \in \mathbb{R}^d$. One can show that $\hat{\xi} \in \hat{\mathcal{N}}$ if and only if $\hat{\xi} = \sum_j \delta_{x_j}$ where $\{x_j\} \subset \mathbb{R}^d$ is a locally finite set. Trivially, a simple counting measure $\hat{\xi}$ can be identified with its support. Given $x \in \mathbb{R}^d$ the translated counting measure $S_x \hat{\xi}$ is defined as $S_x \hat{\xi} = \sum_j \delta_{x_j - x}$ if $\hat{\xi} = \sum_j \delta_{x_j}$.

The space $\hat{\mathcal{N}}$ is endowed with the σ -algebra of measurable subsets generated by the maps

$$\hat{\mathcal{N}} \ni \hat{\xi} \rightarrow \hat{\xi}(B) \in \mathbb{N}, \quad B \in \mathcal{B}(\mathbb{R}^d) \text{ bounded}.$$

A *simple point process* (on \mathbb{R}^d) is a measurable map Φ from a probability space into $\hat{\mathcal{N}}$. With abuse of notation, we identify a simple point process with its distribution $\hat{\mathcal{P}}$ on $\hat{\mathcal{N}}$. Moreover, one calls it *stationary* if $\hat{\mathcal{P}}(A) = \hat{\mathcal{P}}(S_x A)$, for all $x \in \mathbb{R}^d$ and $A \subset \hat{\mathcal{N}}$ measurable. In this case, we define the κ -moment ρ_κ as

$$\rho_\kappa := \mathbf{E}_{\hat{\mathcal{P}}} \left(\hat{\xi}([0, 1]^d)^\kappa \right), \quad \kappa > 0. \quad (2.1)$$

Then $\rho = \rho_1$ is the so-called *intensity* of the process.

The *Palm distribution* $\hat{\mathcal{P}}_0$ associated to a stationary simple point process $\hat{\mathcal{P}}$ on \mathbb{R}^d with finite intensity ρ is the probability measure on the measurable subset $\hat{\mathcal{N}}_0 \subset \hat{\mathcal{N}}$,

$$\hat{\mathcal{N}}_0 := \left\{ \hat{\xi} \in \hat{\mathcal{N}} : \hat{\xi}(0) = 1 \right\},$$

characterized by the Campbell identity:

$$\hat{\mathcal{P}}_0(A) = \frac{1}{\rho K^d} \int_{\hat{\mathcal{N}}} \hat{\mathcal{P}}(d\hat{\xi}) \int_{Q_K} \hat{\xi}(dx) \chi_A(S_x \hat{\xi}), \quad \forall A \subset \hat{\mathcal{N}}_0 \text{ measurable}, \quad (2.2)$$

where $Q_K = [-K/2, K/2]^d$, $K > 0$ (since $\hat{\mathcal{P}}$ is stationary, the r.h.s. in Campbell identity does not depend on K). As discussed in [6], the point process $\hat{\mathcal{P}}_0$ can be thought of as obtained from the point process $\hat{\mathcal{P}}$ by conditioning the latter to give positive mass at the origin.

Given two simple point processes $\hat{\mathcal{P}}, \hat{\mathcal{P}}'$ one says that $\hat{\mathcal{P}}$ is *stochastically dominated* by $\hat{\mathcal{P}}'$, shortly $\hat{\mathcal{P}} \preceq \hat{\mathcal{P}}'$, if there exists a coupling of $\hat{\mathcal{P}}, \hat{\mathcal{P}}'$ such that almost surely $\hat{\xi} \subset \hat{\xi}'$, with $(\hat{\xi}, \hat{\xi}')$ denoting the random sets with marginal distributions given by $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ respectively. We refer to [9] for more details on stochastic domination between point processes.

We denote \mathcal{N} the space of (marked) simple counting measures ξ on $\mathbb{R}^d \times [-1, 1]$ such that $\xi(B \times [-1, 1]) < \infty$ for all $B \in \mathcal{B}(\mathbb{R}^d)$ bounded, and $\xi(\{x\} \times [-1, 1]) \in \{0, 1\}$ for all $x \in \mathbb{R}^d$. One can show that $\xi \in \mathcal{N}$ if and only if $\xi = \sum_j \delta_{(x_j, E_j)}$, where $\{(x_j, E_j)\} \subset \mathbb{R}^d \times [-1, 1]$

is a locally finite set such that for each $x \in \mathbb{R}$ there is at most one couple (x_j, E_j) with $x_j = x$. Trivially, a simple counting measure ξ can be identified with its support. The value E_j is called the *mark* at x_j . For physical reasons, we call it the *energy mark*. Given $\xi \in \mathcal{N}$ we write $\hat{\xi}$ for the simple counting measure on \mathbb{R}^d defined as $\hat{\xi}(B) = \xi(B \times [-1, 1])$, for all $B \in \mathcal{B}(\mathbb{R}^d)$ bounded. Given $x \in \mathbb{R}^d$ the translated simple counting measure $S_x \xi$ is defined as $S_x \xi = \sum_j \delta_{(x_j - x, E_j)}$ if $\xi = \sum_j \delta_{(x_j, E_j)}$.

The space \mathcal{N} is endowed with the σ -algebra of measurable subsets generated by the maps

$$\mathcal{N} \ni \xi \rightarrow \xi(B) \in \mathbb{N}, \quad B \in \mathcal{B}(\mathbb{R}^d \times [-1, 1]) \text{ bounded.}$$

A *marked simple point process* on \mathbb{R}^d is a measurable map Φ from a probability space into \mathcal{N} . Again, with abuse of notation, we will identify it with its distribution \mathcal{P} on \mathcal{N} . Moreover, it is called *stationary* if $\mathcal{P}(A) = \mathcal{P}(S_x A)$, for all $x \in \mathbb{R}^d$ and $A \subset \mathcal{N}$ measurable. In this case, we define the κ -moment ρ_κ as

$$\rho_\kappa := \mathbf{E}_{\mathcal{P}} \left(\hat{\xi}([0, 1]^d)^\kappa \right), \quad \kappa > 0,$$

and call $\rho := \rho_1$ the intensity of the process.

The *Palm distribution* \mathcal{P}_0 associated to a stationary marked simple point process \mathcal{P} with finite intensity ρ is the probability measure on the measurable subset $\mathcal{N}_0 \subset \mathcal{N}$,

$$\mathcal{N}_0 := \left\{ \xi \in \mathcal{N} : \hat{\xi}(0) = 1 \right\},$$

characterized by the Campbell identity

$$\mathcal{P}_0(A) = \frac{1}{\rho K^d} \int_{\mathcal{N}} \mathcal{P}(d\xi) \int_{Q_K} \hat{\xi}(dx) \chi_A(S_x \xi), \quad \forall A \subset \mathcal{N}_0 \text{ measurable}, K > 0. \quad (2.3)$$

A standard procedure for obtaining a marked simple point process from a given simple point process on \mathbb{R}^d is the ν -randomization, where ν is a probability measure on $[-1, 1]$: given a realization of the simple point process on \mathbb{R}^d , its points are marked by i.i.d. random variables with common law ν . It is simple to check that the ν -randomization of the Palm distribution associated to a given stationary simple point process coincides with the Palm distribution associated to the ν -randomization of the stationary simple point process.

We conclude this section recalling some technical results derived in [8]. In particular, point (i) of the following lemma follows from [8][Lemma 1, (i)] and the Monotone Convergence Theorem, while the proof of point (ii) is similar to the proof of [8][Lemma 2]:

Lemma 1. [8] *Let \mathcal{P}_0 be the Palm distribution associated to a stationary marked simple point process.*

(i) *Let $f : \mathcal{N}_0 \times \mathcal{N}_0 \rightarrow \mathbb{R}$ be a measurable function which is non negative or such that $\int \hat{\xi}(dx) |f(\xi, S_x \xi)|$ and $\int \hat{\xi}(dx) |f(S_x \xi, \xi)|$ are in $L^1(\mathcal{P}_0)$. Then*

$$\int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx) f(\xi, S_x \xi) = \int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx) f(S_x \xi, \xi).$$

(ii) *Let n be a nonnegative integer such that $\rho_{n+1} < \infty$. Then*

$$\int \mathcal{P}_0(d\xi) \left(\int \hat{\xi}(dx) e^{-\gamma|x|} \right)^n < \infty$$

for any $\gamma > 0$.

3. UPPER BOUNDS VIA SPECIAL TEST FUNCTIONS

In this section we let \mathcal{P}_0 be the Palm distribution associated to the ν -randomization of a stationary simple point process with finite intensity and obtain upper bounds on $D(\beta)$ by choosing special test functions $f \in L^\infty(\mathcal{P}_0)$ in the r.h.s. of (1.12). We suppose that ν is not concentrated in a unique value, i.e. ν is not of the form $\nu = \delta_E$. This implies that for \mathcal{P}_0 -a.a. ξ , $S_x \xi \neq S_y \xi$ if $x, y \in \hat{\xi}$ and $x \neq y$. Note that the above assumption is satisfied whenever (1.15) is fulfilled, moreover in the case $\nu = \delta_E$ the β -dependence of the diffusion matrix is trivial.

Given $\xi \in \mathcal{N}_0$, let $\mathcal{E}^\beta(\xi)$ be a family of non oriented links in $\hat{\xi}$, i.e.

$$\mathcal{E}^\beta(\xi) \subset \left\{ \{x, y\} : x, y \in \hat{\xi} \text{ and } x \neq y \right\}, \quad (3.1)$$

covariant w.r.t. space translations, i.e.

$$\mathcal{E}^\beta(S_x \xi) = \mathcal{E}^\beta(\xi) - x, \quad \forall \xi \in \mathcal{N}_0, x \in \hat{\xi}. \quad (3.2)$$

Consider the graph $\mathcal{G}^\beta(\xi)$ with vertexes set $\mathcal{V}^\beta(\xi)$ and edges set $\mathcal{E}^\beta(\xi)$ where

$$\mathcal{V}^\beta(\xi) = \left\{ x \in \hat{\xi} : \exists y \in \hat{\xi} \text{ with } \{x, y\} \in \mathcal{E}^\beta(\xi) \right\}. \quad (3.3)$$

Given x, y in $\mathcal{V}^\beta(\xi)$ we say that they are connected if there exists a path in $\mathcal{G}^\beta(\xi)$ going from x to y . Moreover, we denote by $C_x^\beta(\xi)$ the connected component in $\mathcal{G}^\beta(\xi)$ containing x if $x \in \mathcal{V}^\beta(\xi)$, while we set $C_x^\beta(\xi) = \emptyset$ if $x \in \hat{\xi} \setminus \mathcal{V}^\beta(\xi)$.

Proposition 1. *Let \mathcal{P}_0 be the Palm distribution associated to the ν -randomization of a stationary simple point process with $\rho_2 < \infty$ and $\nu \neq \delta_E$ for any $E \in [-1, 1]$. Suppose that for each $\beta > 0$ a random graph $\mathcal{G}^\beta(\xi) = (\mathcal{V}^\beta(\xi), \mathcal{E}^\beta(\xi))$, satisfying (3.1), (3.2) and (3.3), is assigned and that the following assumptions (A1), (A2) are fulfilled:*

- (A1) *There exists a positive function $\ell(\beta)$ such that*

$$|x - y| > \ell(\beta) \Rightarrow \{x, y\} \notin \mathcal{E}^\beta(\xi), \quad \forall x, y \in \hat{\xi}, \forall \xi \in \mathcal{N}_0; \quad (3.4)$$

- (A2) *The function $\mathcal{N}_0 \ni \xi \rightarrow C_0^\beta(\xi) \in \mathcal{N}$ is measurable and for some $\varepsilon > 0$*

$$\limsup_{\beta \uparrow \infty} \mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^{2+\varepsilon} \right) < \infty, \quad (3.5)$$

$$\rho_{\lceil 2(1+\varepsilon)/\varepsilon \rceil + 1} < \infty, \quad (3.6)$$

where $\lceil a \rceil$ denotes the smallest integer larger than a .

Then, for all $i = 1, \dots, d$ and β large enough,

$$D_{i,i}(\beta) \leq 6 \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left((x^{(i)})^2 + \ell(\beta)^2 |C_0^\beta(\xi)|^2 \right) \mathbb{I}_{x \notin C_0^\beta(\xi)}, \quad (3.7)$$

where $x^{(i)}$ denotes the i -th coordinate of x and $\mathbb{I}_{x \notin C_0^\beta(\xi)}$ is the characteristic function of the event $\{x \notin C_0^\beta(\xi)\}$.

Moreover, suppose that the following additional assumption is fulfilled:

- (A3) *There exists a positive function $C(\beta)$ such that*

$$\{x, y\} \notin \mathcal{E}^\beta(\xi) \Rightarrow c_{x,y}(\xi) \leq C(\beta), \quad \forall x, y \in \hat{\xi}, \forall \xi \in \mathcal{N}_0. \quad (3.8)$$

Then, for all $i = 1, \dots, d$, $\kappa \in (0, 1)$ and β large enough,

$$D_{i,i}(\beta) \leq c(\kappa) C(\beta)^\kappa (1 + \ell(\beta)^2), \quad (3.9)$$

for a suitable positive constant $c(\kappa)$ depending on κ , but not on β . In particular,

$$D(\beta) \leq d c(\kappa) C(\beta)^\kappa (1 + \ell(\beta)^2) \mathbb{I}. \quad (3.10)$$

The proof of the above Proposition is obtained by plugging suitable test functions f in the variational formula (1.12). Our test functions are similar to the ones used in [16][Proof of Theorem 3.12].

Remark 1. As one can easily deduce from the proof of the above Proposition, the results (3.7), (3.9) and (3.10) hold for all $\beta > 0$ if condition (3.5) is satisfied with $\limsup_{\beta \uparrow \infty}$ replaced by $\sup_{\beta > 0}$.

Proof. Assume (A1), (A2) to be satisfied and, given a positive integer N , consider the test function $f_N^\beta : \mathcal{N}_0 \rightarrow \mathbb{R}_{\geq 0}$ defined as follows:

$$f_N^\beta(\xi) = \begin{cases} -\min \left\{ z^{(i)} : z \in C_0^\beta(\xi) \right\} & \text{if } 1 \leq |C_0^\beta(\xi)| \leq N, \\ 0 & \text{otherwise,} \end{cases} \quad (3.11)$$

where $z^{(i)}$ denotes the i -th coordinate of z . Due to (A2) f_N^β is measurable, while due to (A1)

$$0 \leq f_N^\beta(\xi) \leq |C_0^\beta(\xi)| \ell(\beta), \quad \forall \xi \in \mathcal{N}_0. \quad (3.12)$$

In particular,

$$\left(x^{(i)} - \nabla_x f_N^\beta(\xi) \right)^2 \leq 3 \left((x^{(i)})^2 + \ell(\beta)^2 |C_0^\beta(\xi)|^2 + \ell(\beta)^2 |C_0^\beta(S_x \xi)|^2 \right), \quad \forall \xi \in \mathcal{N}_0. \quad (3.13)$$

Due to (1.12)

$$D_{i,i}(\beta) \leq \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left(x^{(i)} - \nabla_x f_N^\beta(\xi) \right)^2 = I_N^{(1)}(\beta) + I_N^{(2)}(\beta) + I_N^{(3)}(\beta), \quad (3.14)$$

where

$$\begin{aligned} I_N^{(1)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left(x^{(i)} - \nabla_x f_N^\beta(\xi) \right)^2 \mathbb{I}_{x \in C_0^\beta(\xi)} \mathbb{I}_{|C_0^\beta(\xi)| \leq N}, \\ I_N^{(2)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left(x^{(i)} - \nabla_x f_N^\beta(\xi) \right)^2 \mathbb{I}_{x \in C_0^\beta(\xi)} \mathbb{I}_{|C_0^\beta(\xi)| > N}, \\ I_N^{(3)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left(x^{(i)} - \nabla_x f_N^\beta(\xi) \right)^2 \mathbb{I}_{x \notin C_0^\beta(\xi)}, \end{aligned}$$

where \mathbb{I}_A denotes that characteristic function of the event A .

• *Estimate of $I_N^{(1)}(\beta)$.* If $x \in C_0^\beta(\xi)$ and $|C_0^\beta(\xi)| \leq N$, then

$$C_0^\beta(S_x \xi) = C_0^\beta(\xi) - x \quad (3.15)$$

and in particular

$$|C_0^\beta(S_x \xi)| = |C_0^\beta(\xi)| \in [1, N]. \quad (3.16)$$

Due to (3.15) and (3.16) we get $f_N^\beta(S_x \xi) = x^{(i)} + f_N^\beta(\xi)$ and therefore $x^{(i)} - \nabla_x f_N^\beta(\xi) = 0$, thus implying

$$I_N^{(1)}(\beta) = 0. \quad (3.17)$$

- *Estimate of $I_N^{(2)}(\beta)$.* We claim that, if β is large enough,

$$\lim_{N \uparrow \infty} I_N^{(2)}(\beta) = 0. \quad (3.18)$$

In order to prove the above limit we observe that, due to (3.13) and since $c_{0,x}(\xi) \leq e^{-|x|}$,

$$I_N^{(2)}(\beta) \leq 3 \left(J_N^{(1)}(\beta) + \ell(\beta)^2 J_N^{(2)}(\beta) + \ell(\beta)^2 J_N^{(3)}(\beta) \right), \quad (3.19)$$

where

$$\begin{aligned} J_N^{(1)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |x|^2 \mathbb{I}_{|C_0^\beta(\xi)| > N}, \\ J_N^{(2)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)} \mathbb{I}_{|C_0^\beta(\xi)| > N}, \\ J_N^{(3)}(\beta) &= \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |C_0^\beta(S_x \xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)} \mathbb{I}_{|C_0^\beta(\xi)| > N}. \end{aligned}$$

Due to (A2), $\lim_{N \uparrow \infty} \mathbb{I}_{|C_0^\beta(\xi)| > N} = 0$ for \mathcal{P}_0 -a.a. ξ . Hence, by the Dominated Convergence Theorem, we only need to prove that

$$\int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |x|^2 < \infty, \quad (3.20)$$

$$\int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)} < \infty, \quad (3.21)$$

$$\int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |C_0^\beta(S_x \xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)} < \infty, \quad (3.22)$$

for β large enough.

Since $\rho_2 < \infty$, Lemma 1 (ii) implies (3.20). Since

$$\text{l.h.s. of (3.21)} \leq \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-|x|} |C_0^\beta(\xi)|^2,$$

given $p, q > 1$ with $1/p + 1/q = 1$, due to Hölder inequality

$$\text{l.h.s. of (3.21)} \leq \mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^{2q} \right)^{1/q} \mathbf{E}_{\mathcal{P}_0} \left(\left(\int \hat{\xi}(dx) e^{-|x|} \right)^p \right)^{1/p}.$$

Choosing $2q = 2 + \varepsilon$ (hence $p = (2 + \varepsilon)/\varepsilon$) we have that, due to (3.5), (3.6) and Lemma 1 (ii), both the factors in the r.h.s. are finite for β large enough. Hence (3.21) is true. Finally we observe that the l.h.s. of (3.21) equals the l.h.s. of (3.22) due to Lemma 1 (i). In fact, consider the function f defined on $\mathcal{N}_0 \times \mathcal{N}_0$ as

$$f(\xi, \zeta) = \begin{cases} e^{-|x|} |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)} & \text{if } \zeta = S_x \xi, \ x \in \hat{\xi}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the above definition is well posed \mathcal{P}_0 -a.s. since due to the choice $\nu \neq \delta_E$ we have that $S_x \xi \neq S_y \xi$ if $x, y \in \hat{\xi}$, $x \neq y$, for \mathcal{P}_0 -a.a. ξ . Then, if $x \in \hat{\xi}$,

$$\begin{aligned} f(\xi, S_x \xi) &= e^{-|x|} |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)}, \\ f(S_x \xi, \xi) &= e^{-|x|} |C_0^\beta(S_x \xi)|^2 \mathbb{I}_{-x \in C_0^\beta(S_x \xi)} = e^{-|x|} |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)}. \end{aligned}$$

Note that in the last identity we have used (3.2) which implies that

$$|C_0^\beta(S_x \xi)|^2 \mathbb{I}_{-x \in C_0^\beta(S_x \xi)} = |C_0^\beta(\xi)|^2 \mathbb{I}_{x \in C_0^\beta(\xi)}.$$

Hence, due to Lemma 1 (i) we conclude that the l.h.s. of (3.21) equals the l.h.s. of (3.22).

• *Estimate of $I_N^{(3)}(\beta)$.* Due to (3.13) we can bound

$$I_N^{(3)}(\beta) \leq 3 \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left((x^{(i)})^2 + \ell(\beta)^2 |C_0^\beta(\xi)|^2 + \ell(\beta)^2 |C_0^\beta(S_x \xi)|^2 \right) \mathbb{I}_{x \notin C_0^\beta(\xi)}. \quad (3.23)$$

By defining now $f(\xi, \zeta)$ as

$$f(\xi, \zeta) = \begin{cases} c_{0,x}(\xi) |C_0^\beta(\xi)|^2 \mathbb{I}_{x \notin C_0^\beta(\xi)} & \text{if } \zeta = S_x \xi, x \in \hat{\xi}, \\ 0 & \text{otherwise} \end{cases}$$

and reasoning as in the proof that the l.h.s. of (3.21) equals the l.h.s. of (3.22), it is simple to show that

$$\int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) |C_0^\beta(\xi)|^2 \mathbb{I}_{x \notin C_0^\beta(\xi)} = \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) |C_0^\beta(S_x \xi)|^2 \mathbb{I}_{x \notin C_0^\beta(\xi)}.$$

Hence

$$I_N^{(3)}(\beta) \leq 6 \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) c_{0,x}(\xi) \left((x^{(i)})^2 + \ell(\beta)^2 |C_0^\beta(\xi)|^2 \right) \mathbb{I}_{x \notin C_0^\beta(\xi)}. \quad (3.24)$$

• *Conclusions.* The bound (3.7) follows from (3.14), (3.17), (3.18) and (3.24). Suppose now that also assumption (A3) is valid. In particular, if $x \notin C_0^\beta(\xi)$ then $\{0, x\} \notin \mathcal{E}^\beta(\xi)$ and therefore $c_{0,x}(\xi) \leq C(\beta)$. In particular $c_{0,x}(\xi) \leq C(\beta)^\kappa e^{-(1-\kappa)|x|}$, for all $\kappa \in (0, 1)$. Due to (3.7) we get

$$\begin{aligned} D_{i,i}(\beta, \gamma) &\leq 6 C(\beta)^\kappa \int \mathcal{P}_0(\xi) \int \hat{\xi}(dx) e^{-(1-\kappa)|x|} \left(|x|^2 + \ell(\beta)^2 |C_0^\beta(\xi)|^2 \right) \\ &\leq C(\beta)^\kappa c(\kappa) (1 + \ell(\beta)^2), \end{aligned} \quad (3.25)$$

where the last bound follows from (3.5) and the same arguments used for proving (3.21). Hence the proof of (3.9) is concluded.

Finally, let us prove (3.10). The matrix $D(\beta)$ is positive and symmetric. In particular,

$$(a, D(\beta)a) \leq \max\{\lambda_i : 1 \leq i \leq d\} (a, a), \quad \forall a \in \mathbb{R}^d,$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $D(\beta)$. Since $\lambda_i \geq 0$ for each i ,

$$(a, D(\beta)a) \leq \text{Tr}(D(\beta)) (a, a) = \left(\sum_{i=1}^d D_{i,i}(\beta) \right) (a, a), \quad \forall a \in \mathbb{R}^d,$$

and (3.10) follows from (3.9). \square

4. PROOF OF THEOREM 1 IN THE POISSONIAN CASE

In this section we prove Theorem 1 in the special case that \mathcal{P}_0 is the Palm distribution associated to the ν -randomization of the Poisson point process with intensity $\rho > 0$. In the next section, we extend the proof to the general case. We write $\hat{\mathcal{P}}_\rho$ for the Poisson point process with density ρ , \mathcal{P}_ρ for its ν -randomization, $\hat{\mathcal{P}}_{0,\rho}$ for the Palm distribution associated to $\hat{\mathcal{P}}_\rho$ and $\mathcal{P}_{0,\rho}$ for the Palm distribution associated to \mathcal{P}_ρ . Equivalently, $\mathcal{P}_{0,\rho}$ is the ν -randomization of $\hat{\mathcal{P}}_{0,\rho}$. In what follows, we write Q_L for the cube $[-L/2, L/2]^d$.

The proof is based on Proposition 1, scaling arguments and continuum percolation. We recall some results of continuum percolation referring to [11] for a more detailed discussion. Given $r > 0$ we write $B_r(x)$ for the closed ball centered at $x \in \mathbb{R}^d$ with radius r . If $x = 0$ we simply write B_r . We define the *occupied region* of the Boolean model with radius r driven by $\hat{\xi} \in \hat{\mathcal{N}}$ as

$$\mathbb{X}_r(\hat{\xi}) := \cup_{x \in \hat{\xi}} B_r(x) = \{y \in \mathbb{R}^d : d(y, \hat{\xi}) \leq r\},$$

where $d(\cdot, \cdot)$ denotes the euclidean distance. The connected components in the occupied region will be called *occupied components*. For $A \subset \mathbb{R}^d$, we denote by $W_r(A) = W_r(A)[\hat{\xi}]$ the union of all occupied components having non-empty intersection with A . Given $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ we write $A \xleftrightarrow{r} B$ if there exists a path inside $\mathbb{X}_r(\hat{\xi})$ connecting A and B .

The following results hold for stationary Poisson point processes [11]: there exists a positive density ρ_c such that for all bounded subsets $A \subset \mathbb{R}^d$, $A \neq \emptyset$,

$$\rho_c = \inf \left\{ \rho > 0 : \hat{\mathcal{P}}_\rho [\text{diam } W_1(A) = \infty] > 0 \right\}, \quad (4.1)$$

$$\hat{\mathcal{P}}_\rho \left(A \xleftrightarrow{1} \partial Q_L \right) \leq e^{-c(\rho, A)L}, \quad \forall L > 0, \quad \forall \rho < \rho_c \quad (4.2)$$

for a suitable positive constant $c(\rho, A)$ depending on ρ, A . Above ∂Q_L denotes the border of the cube Q_L . Moreover, one can prove that all occupied components in \mathbb{X}_1 are bounded $\hat{\mathcal{P}}_\rho$ -a.s. for $\rho < \rho_c$, while there exists a unique unbounded occupied component in \mathbb{X}_1 $\hat{\mathcal{P}}_\rho$ -a.s. for $\rho > \rho_c$.

Note that the function $\mathbb{R}^d \ni x \rightarrow x/r \in \mathbb{R}^d$ maps $\hat{\mathcal{P}}_\rho$ in $\hat{\mathcal{P}}_{\rho/r^d}$, namely if $\hat{\xi}$ has law $\hat{\mathcal{P}}_\rho$ then $\{x/r : x \in \hat{\xi}\}$ has law $\hat{\mathcal{P}}_{\rho/r^d}$. This scaling property allows to restate the above results for a fixed density ρ and varying radius r : the positive constant

$$r_c(\rho) := (\rho_c/\rho)^{1/d} = \rho^{-1/d} r_c(1) \quad (4.3)$$

satisfies

$$r_c(\rho) = \inf \left\{ r > 0 : \hat{\mathcal{P}}_\rho [\text{diam } W_r(A) = \infty] > 0 \right\}, \quad (4.4)$$

$$\hat{\mathcal{P}}_\rho \left(A \xleftrightarrow{r} \partial Q_L \right) \leq e^{-c(r, \rho, A)L}, \quad \forall L > 0, \quad \forall r < r_c(\rho), \quad (4.5)$$

for all bounded subsets $A \subset \mathbb{R}^d$, $A \neq \emptyset$, and for a suitable positive constant $c(r, \rho, A)$ depending on r, ρ, A . Moreover, all occupied components in \mathbb{X}_r are bounded $\hat{\mathcal{P}}_\rho$ -a.s. for $r < r_c$, while there exists a unique unbounded occupied component in \mathbb{X}_r $\hat{\mathcal{P}}_\rho$ -a.s. for $r > r_c$.

We point out a simple consequence of (4.5):

Lemma 2. *If $r < r_c(\rho)$ then for all bounded sets $A \subset \mathbb{R}^d$ with $A \neq \emptyset$ and for all $s > 0$*

$$\mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} \left(W_r(A)[\hat{\xi}] \right)^s \right) < \infty. \quad (4.6)$$

Proof. Due to the stationarity of $\hat{\mathcal{P}}_\rho$, we can assume that $0 \in A$ without loss of generality. Fixed $p, q > 1$ with $1/p + 1/q = 1$, by Hölder inequality and (4.5) we get

$$\begin{aligned} \mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} \left(W_r(A) [\hat{\xi}] \right)^s \right) &\leq \sum_{L=1}^{\infty} \mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} (Q_L)^s \mathbb{I}_{\{A \xleftrightarrow{r} \partial Q_{L-1}\}} \right) \\ &\leq \sum_{L=1}^{\infty} \mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} (Q_L)^{sp} \right)^{1/p} \hat{\mathcal{P}}_\rho \left(A \xleftrightarrow{r} \partial Q_{L-1} \right)^{1/q} \leq \sum_{L=1}^{\infty} \mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} (Q_L)^{sp} \right)^{1/p} e^{-cL}. \end{aligned} \quad (4.7)$$

The thesis then follows by observing that, since $\hat{\xi}(Q_L)$ is a Poisson random variable with expectation ρL^d ,

$$\mathbf{E}_{\hat{\mathcal{P}}_\rho} \left(\hat{\xi} (Q_L)^n \right) \leq c(\rho) L^{dn}, \quad \forall L > 0, n \in \mathbb{N},$$

thus implying that the last member in (4.7) is summable. \square

We can now give the proof of Theorem 1 for Poisson point processes:

Proof of Theorem 1 for $\mathcal{P}_0 = \mathcal{P}_{0,\rho}$. Given $\xi \in \mathcal{N}_0$ we set

$$E(\beta) := \beta^{-\frac{d}{\alpha+1+d}},$$

$$\rho(\beta) := \rho \nu([-E(\beta), E(\beta)]) ,$$

$$\ell(\beta) := r_c(\rho(\beta)) = \rho(\beta)^{-1/d} r_c(1),$$

$$\mathcal{E}^\beta(\xi) := \left\{ \{x, y\} : x, y \in \hat{\xi}, x \neq y, |E_x| \leq E(\beta), |E_y| \leq E(\beta) \text{ and } |x - y| \leq \ell(\beta) \right\}.$$

We point out that we could have defined $\ell(\beta) = \gamma r_c(\rho(\beta))$ for an arbitrary $\gamma \in (0, 2)$. Here $\gamma := 1$. Assumption (1.15) implies that

$$0 < \rho(\beta) \leq c_0 \rho \beta^{-\frac{d(\alpha+1)}{\alpha+1+d}}, \quad (4.8)$$

hence

$$\ell(\beta) \geq (c_0 \rho)^{-1/d} r_c(1) \beta^{\frac{\alpha+1}{\alpha+1+d}}. \quad (4.9)$$

In particular, due to (1.8),

$$\begin{aligned} \{x, y\} &\notin \mathcal{E}^\beta(\xi) \\ \Rightarrow c_{x,y}(\xi) &\leq \exp \{-\ell(\beta) \wedge [\kappa_1 \beta E(\beta)]\} \leq \exp \left\{ -c(\alpha, \rho) \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} =: C(\beta). \end{aligned} \quad (4.10)$$

Then, due to (3.10), in order to conclude the proof of Theorem 1 it is enough to check that the assumptions of Proposition 1 are fulfilled when the graph $\mathcal{G}^\beta(\xi) = (\mathcal{V}^\beta(\xi), \mathcal{E}^\beta(\xi))$ is defined via (3.3). Conditions (3.1), (3.2) and (3.3) are trivially satisfied. (A1) is obvious, (A3) has already been checked: it therefore only remains to consider (A2). Since Poisson point processes have finite moments, the non trivial condition to be checked is given by (3.5), which can be justified by means of scaling arguments and Lemma 2 as follows.

As discussed in [6], the process $\hat{\xi}$ with law $\hat{\mathcal{P}}_{0,\rho}$ can be constructed by setting $\hat{\xi} := \hat{\omega} \cup \{0\}$, where $\hat{\omega}$ is a Poisson point process with law $\hat{\mathcal{P}}_\rho$. Let ω be the ν -randomization of the process $\hat{\omega}$ and let E_0 be a random variable with law ν , independent from ω . Then $\xi := \omega \cup \{(0, E_0)\}$ has law $\mathcal{P}_{0,\rho}$. Setting

$$\hat{\omega}_\beta = \{x \in \hat{\omega} : |E_x| \leq E(\beta)\}, \quad r(\beta) = \ell(\beta)/2,$$

we get

$$|C_0^\beta(\xi)| \leq 1 + \hat{\omega}_\beta(W_{r(\beta)}(B_{r(\beta)})[\hat{\omega}_\beta]). \quad (4.11)$$

Note that the process $\hat{\omega}_\beta$, obtained by thinning the Poisson process $\hat{\omega}$ with density ρ , has law $\hat{\mathcal{P}}_{\rho(\beta)}$. We now consider the space rescaling

$$\mathbb{R}^d \ni x \rightarrow x \rho(\beta)^{1/d} = x r_c(1)/\ell(\beta) \in \mathbb{R}^d.$$

Since the above function maps $\hat{\mathcal{P}}_{\rho(\beta)}$ onto $\hat{\mathcal{P}}_1$ and points at distance $r(\beta)$ into points at distance $r_c(1)/2$, the random variable

$$\hat{\omega}_\beta \left(W_{r(\beta)} \left(B_{r(\beta)} \right) \left[\hat{\omega}_\beta \right] \right) \quad (4.12)$$

has the same law as

$$\hat{\omega}_* \left(W_{r_c(1)/2} \left(B_{r_c(1)/2} \right) \left[\hat{\omega}_* \right] \right), \quad (4.13)$$

where $\hat{\omega}_*$ has law $\hat{\mathcal{P}}_1$. The random variable (4.13) is β -independent and has finite moments due to Lemma 2. Hence all moments of (4.12) are β -independent and finite. Due to (4.11), (3.5) is satisfied and we can apply Proposition 1. \square

5. PROOF OF THEOREM 1 IN THE GENERAL CASE

We now explain how one can derive Theorem 1 in the general (non Poissonian) case, under the domination assumption (ii).

First we observe that due to assumption (ii) $\hat{\mathcal{P}}$ has finite moments ρ_κ for all $\kappa \geq 0$. In fact, assumption (ii) trivially implies that $\hat{\mathcal{P}}^{(p)}$ has finite moments. Hence, denoting by X_n a generic binomial variable with parameters n, p , we have

$$\begin{aligned} \infty > \mathbf{E}_{\hat{\mathcal{P}}^{(p)}} \left(\hat{\xi}([0, 1]^d)^\kappa \right) &= \sum_{n=1}^{\infty} \hat{\mathcal{P}} \left(\hat{\xi}([0, 1]^d) = n \right) \sum_{j=0}^n j^\kappa \binom{n}{j} p^j (1-p)^{n-j} = \\ &= \sum_{n=1}^{\infty} \hat{\mathcal{P}} \left(\hat{\xi}([0, 1]^d) = n \right) \mathbf{E}(X_n^\kappa). \end{aligned}$$

Since for any positive integer κ we have $\mathbf{E}(X_n^\kappa) = c(\kappa)n^\kappa$, this implies that

$$\rho_\kappa = \sum_{n=1}^{\infty} \hat{\mathcal{P}} \left(\hat{\xi}([0, 1]^d) = n \right) n^\kappa < \infty, \quad \forall \kappa \in \mathbb{N}.$$

Define

$$E(\beta) := \beta^{-\frac{d}{\alpha+1+d}}. \quad (5.1)$$

Since $E(\beta) \downarrow 0$ as $\beta \uparrow \infty$, due to (1.15) we can find β_* such that

$$\gamma := \nu([-E(\beta_*), E(\beta_*)]) \leq p.$$

Since the γ -thinning $\hat{\mathcal{P}}^{(\gamma)}$ is stochastically dominated by the p -thinning $\hat{\mathcal{P}}^{(p)}$, assumption (ii) in Theorem 1 remains valid with p replaced by γ . Hence, without loss of generality, we can assume that $\gamma = p$ in assumption (ii), i.e.

$$p = \nu([-E(\beta_*), E(\beta_*)]) \quad (5.2)$$

for some β_* .

In what follows we take $\beta \geq \beta_*$ and define

$$\ell(\beta) := \lambda \beta^{\frac{\alpha+1}{\alpha+1+d}}, \quad (5.3)$$

$$\mathcal{E}^\beta(\xi) := \left\{ \{x, y\} : x, y \in \hat{\xi}, x \neq y, |E_x| \leq E(\beta), |E_y| \leq E(\beta) \text{ and } |x - y| \leq \ell(\beta) \right\}, \quad (5.4)$$

where the positive β -independent constant λ will be fixed at the end.

We want to apply Proposition 1 where, given $\mathcal{E}^\beta(\xi)$, the graph $\mathcal{G}^\beta(\xi) = (\mathcal{V}^\beta(\xi), \mathcal{E}^\beta(\xi))$ is defined by (3.3). Trivially, the set $\mathcal{E}^\beta(\xi)$ satisfies (3.1) and (3.2), and condition (A1) is fulfilled. Moreover, due to (1.8),

$$\{x, y\} \notin \mathcal{E}^\beta(\xi) \Rightarrow c_{x,y}(\xi) \leq \exp \{-\ell(\beta) \wedge [\kappa_1 \beta E(\beta)]\} \leq \exp \left\{ -c \beta^{\frac{\alpha+1}{\alpha+1+d}} \right\} =: C(\beta),$$

for some β -independent positive constant c . Therefore, condition (A3) is satisfied. Moreover, as already observed, $\rho_\kappa < \infty$ for all $\kappa > 0$. Hence, in order to obtain the bound (3.10), which corresponds to (1.17), we only need to verify (3.5). We will prove that

$$\limsup_{\beta \uparrow \infty} \mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^3 \right) < \infty. \quad (5.5)$$

Due to Campbell identity (2.3), which holds also with Q_K replaced by $\Lambda_K(0)$ [6], we can write

$$\mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^3 \right) = \frac{1}{\rho K^d} \mathbf{E}_{\mathcal{P}} \left(\int_{\Lambda_K(0)} \hat{\xi}(dx) |C_0^\beta(S_x \xi)|^3 \right),$$

where \mathcal{P} denotes the ν -randomization of $\hat{\mathcal{P}}$.

Let us define the conditioned measure

$$\nu_* := \nu(\cdot | |E_0| \leq E(\beta_*)).$$

Then

$$\nu([-E(\beta), E(\beta)]) = p \nu_*([-E(\beta), E(\beta)]).$$

Hence, for $\beta \geq \beta_*$, the random set $\{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$ with ξ chosen with law \mathcal{P} and the random set $\{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$ with ξ chosen with law \mathcal{P}_* , defined as the ν_* -randomization of $\mathcal{P}^{(p)}$, have the same distribution. Since the graph $\mathcal{G}^\beta(\xi) = (\mathcal{V}^\beta(\xi), \mathcal{E}^\beta(\xi))$ is univocally determined by the set $\{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$, we conclude that

$$\mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^3 \right) = \frac{1}{\rho K^d} \mathbf{E}_{\mathcal{P}_*} \left(\int_{\Lambda_K(0)} \hat{\xi}(dx) |C_0^\beta(S_x \xi)|^3 \right). \quad (5.6)$$

In order to bound the r.h.s. of (5.6) using the domination assumption (ii), we consider the partition of \mathbb{R}^d in the cubes $\Lambda_K(x)$, $x \in K\mathbb{Z}^d$, and to each $A \subset \mathbb{R}^d$ we associate the sets

$$V_K(A) := \{x \in K\mathbb{Z}^d : \Lambda_K(x) \cap A \neq \emptyset\},$$

$$E_K(A) := \left\{ \{x, y\} : x, y \in V_K(A), x \neq y, |x - y| \leq \ell(\beta) + d\sqrt{K} \right\}.$$

We define $S_K(A)$ as the connected cluster in the graph $G_K(A) = (V_K(A), E_K(A))$ containing the origin if $0 \in V_K(A)$, and as the empty set if $0 \notin V_K(A)$. Finally, we set

$$C_K(A) := \cup_{x \in S_K(A)} (\Lambda_K(x) \cap A).$$

If $A = \{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$ we will simply write $\mathcal{C}_K^\beta(\xi)$ for $C_K(A)$.

Lemma 3. *For all $x \in \Lambda_K(0)$, it holds*

$$C_0^\beta(S_x\xi) + x \subset (\xi \cap \Lambda_K(0)) \cup \mathcal{C}_K^\beta(\xi). \quad (5.7)$$

Proof. Due to the covariant property (3.2), $C_0^\beta(S_x\xi) + x = C_x^\beta(\xi)$. If this set is empty the thesis is trivially true. Otherwise suppose that $z \in C_x^\beta(\xi)$. If $z \in \Lambda_K(0)$ then z belongs to the r.h.s. of (5.7). If $z \in \Lambda_K(u)$ for some $u \in K\mathbb{Z}^d \setminus \{0\}$, by following the path connecting x to z in $\mathcal{G}^\beta(\xi)$ we can define a sequence of distinct points $u_0 = 0, u_1, u_2, \dots, u_{n-1}, u_n = u$ in $K\mathbb{Z}^d$ such that for each i , $0 \leq i \leq n-1$, there exist points $a_i \in \Lambda_K(u_i)$, $b_i \in \Lambda_K(u_{i+1})$ with $\{a_i, b_i\} \in \mathcal{E}^\beta(\xi)$. Hence $\{u_i, u_{i+1}\} \in E_K(A)$ for all $0 \leq i \leq n-1$, where $A = \{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$. This proves that 0 and u are connected in the graph $G_K(A)$. In particular, $u \in S_K(A)$ and therefore $z \in \Lambda_K(u) \cap A \subset \mathcal{C}_K^\beta(\xi)$. \square

Due to the above lemma we get the bound

$$\begin{aligned} \text{r.h.s. of (5.6)} &\leq \frac{c}{\rho K^d} \mathbf{E}_{\mathcal{P}_*} \left(\int_{\Lambda_K(0)} \hat{\xi}(dx) \left(\hat{\xi}(\Lambda_K(0))^3 + |\mathcal{C}_K^\beta(\xi)|^3 \right) \right) = \\ &= c p \mathbf{E}_{\mathcal{P}_*} \left(\hat{\xi}(\Lambda_K(0))^3 \right) + c p \mathbf{E}_{\mathcal{P}_*} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right). \end{aligned} \quad (5.8)$$

Since

$$\mathbf{E}_{\mathcal{P}_*} \left(\hat{\xi}(\Lambda_K(0))^3 \right) = \mathbf{E}_{\mathcal{P}^{(p)}} \left(\hat{\xi}(\Lambda_K(0))^3 \right) < \infty,$$

in order to conclude the proof we only need to prove that

$$\limsup_{\beta \uparrow \infty} \mathbf{E}_{\mathcal{P}_*} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right) < \infty. \quad (5.9)$$

Let us derive from the domination assumption (ii) that

$$\mathbf{E}_{\mathcal{P}_*} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right) \leq \mathbf{E}_{\mathcal{P}_{*,\rho'}} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right), \quad (5.10)$$

where $\mathcal{P}_{*,\rho'}$ is the ν_* -randomization of the Poisson point process $\hat{\mathcal{P}}_{\rho'}$. To this aim, we define

$$\Phi_K(\hat{\xi}) = \left\{ \hat{\xi}(\Lambda_K(x)) : x \in K\mathbb{Z}^d \right\}, \quad \hat{\xi} \in \mathcal{N}.$$

We claim that, given a marked point process \mathcal{Q} obtained as ν_* -randomization of a stationary simple point process, the conditional expectation

$$\mathbf{E}_{\mathcal{Q}} \left(|\mathcal{C}_K^\beta(\xi)|^3 \mid \Phi_K \right) \quad (5.11)$$

is an increasing function in Φ_K that does not depend on \mathcal{Q} . In order to prove this statement, we write $\text{Bin}(N, p)$ for a generic binomial variable with parameters N, p and recall that $\text{Bin}(N, p)$ is stochastically dominated by $\text{Bin}(N', p)$ if $N \leq N'$. Given $\Phi_K(\hat{\xi})$, the random variables $(a_x(\xi), x \in K\mathbb{Z}^d)$ defined as

$$a_x(\xi) = |\{z \in \hat{\xi} \cap \Lambda_K(x) : |E_z| \leq E(\beta)\}|$$

are independent binomial r.v.'s with parameters $\hat{\xi}(\Lambda_K(x)), \nu_*[-E(\beta), E(\beta)]$. In particular, the conditional law of $(a_x(\xi), x \in K\mathbb{Z}^d)$ given Φ_K does not depend on \mathcal{Q} and it increases with Φ_K . Besides, the cardinality $|\mathcal{C}_K^\beta(\xi)|$ is an increasing function of $(a_x(\xi), x \in K\mathbb{Z}^d)$. Since the composition of increasing functions is increasing, we conclude that the conditional expectation (5.11) is an increasing function of Φ_K , independent of \mathcal{Q} . This proves our claim.

Since by assumption (ii) the random field $\Phi_K(\hat{\xi})$ with $\hat{\xi}$ chosen with law $\hat{\mathcal{P}}^{(p)}$ is stochastically dominated by the random field $\Phi_K(\hat{\xi})$ with $\hat{\xi}$ chosen with law $\hat{\mathcal{P}}_{\rho'}$, we obtain that

$$\begin{aligned} \mathbf{E}_{\mathcal{P}_*} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right) &= \mathbf{E}_{\hat{\mathcal{P}}^{(p)}} \left(\mathbf{E}_{\mathcal{P}_*} \left(|\mathcal{C}_K^\beta(\xi)|^3 | \Phi_K(\hat{\xi}) \right) \right) \leq \\ &\mathbf{E}_{\hat{\mathcal{P}}_{\rho'}} \left(\mathbf{E}_{\mathcal{P}_{*,\rho'}} \left(|\mathcal{C}_K^\beta(\xi)|^3 | \Phi_K(\hat{\xi}) \right) \right) = \mathbf{E}_{\mathcal{P}_{*,\rho'}} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right), \end{aligned}$$

thus concluding the proof of (5.10).

Due to (5.10), in order to derive (5.9) and complete the proof of Theorem 1 we only need to show that

$$\limsup_{\beta \uparrow \infty} \mathbf{E}_{\mathcal{P}_{*,\rho'}} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right) < \infty. \quad (5.12)$$

To this aim we will use scaling and percolation arguments as in the previous section.

The random set $A = \{x \in \hat{\xi} : |E_x| \leq E(\beta)\}$, where ξ is chosen with law $\mathcal{P}_{*,\rho'}$, is a Poisson point process with intensity

$$\mu(\beta) = \rho' \nu_*([-E(\beta), E(\beta)]).$$

Hence, by definition of $\mathcal{C}_K^\beta(\xi)$, we have

$$\mathbf{E}_{\mathcal{P}_{*,\rho'}} \left(|\mathcal{C}_K^\beta(\xi)|^3 \right) = \mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left(|C_K(\hat{\xi})|^3 \right). \quad (5.13)$$

Let

$$r(\beta) := 2\sqrt{d}\ell(\beta)/3.$$

Recall that B_s denotes the closed ball of radius s centered at the origin. Using the same notation as in the previous section, for β large enough we can bound

$$\mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left(|C_K(\hat{\xi})|^3 \right) \leq \mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left[\hat{\xi} \left(W_{r(\beta)}(B_{r(\beta)})[\hat{\xi}] \right)^3 \right]. \quad (5.14)$$

By the scaling invariance of the Poisson point process, for each $\gamma > 0$,

$$\mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left[\hat{\xi} \left(W_{r(\beta)}(B_{r(\beta)})[\hat{\xi}] \right)^3 \right] = \mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)\gamma^d}} \left[\hat{\xi} \left(W_{r(\beta)/\gamma}(B_{r(\beta)/\gamma})[\hat{\xi}] \right)^3 \right].$$

Taking

$$\gamma := \mu(\beta)^{-1/d} = (\rho' \nu_*([-E(\beta), E(\beta)]))^{-1/d},$$

we get

$$\mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left[\hat{\xi} \left(W_{r(\beta)}(B_{r(\beta)})[\hat{\xi}] \right)^3 \right] = \mathbf{E}_{\hat{\mathcal{P}}_1} \left[\hat{\xi} \left(W_{r(\beta)/\gamma}(B_{r(\beta)/\gamma})[\hat{\xi}] \right)^3 \right]. \quad (5.15)$$

Due to the definition of ν_* , $E(\beta)$ and assumption (1.15) we get that

$$\gamma \geq c \beta^{\frac{\alpha+1}{\alpha+1+d}}$$

for some positive constant c depending only on p, ρ', d and on the constant c_0 appearing in (1.15). Since $\ell(\beta) = \lambda \beta^{\frac{\alpha+1}{\alpha+1+d}}$, then

$$r(\beta)/\gamma = 2\sqrt{d}\ell(\beta)/(3\gamma) \leq 2\sqrt{d}\ell(\beta)\beta^{-\frac{\alpha+1}{\alpha+1+d}}/(3c) = 2\sqrt{d}\lambda/(3c). \quad (5.16)$$

It is enough to choose λ such that

$$2\sqrt{d}\lambda/(3c) \leq r_c(1)/2.$$

With this choice (5.15) and (5.16) imply that

$$\sup_{\beta > 0} \mathbf{E}_{\hat{\mathcal{P}}_{\mu(\beta)}} \left[\hat{\xi} \left(W_{r(\beta)}(B_{r(\beta)})[\hat{\xi}] \right)^3 \right] \leq \mathbf{E}_{\hat{\mathcal{P}}_1} \left[\hat{\xi} \left(W_{r_c(1)/2}(B_{r_c(1)/2})[\hat{\xi}] \right)^3 \right]. \quad (5.17)$$

The r.h.s. in the above inequality is finite due to Lemma 2. This observation together with (5.13), (5.14) and (5.17) imply (5.12). This concludes the proof of Theorem 1.

Remark 2. *In the case that $\hat{\mathcal{P}}$ is stochastically dominated by the Poisson point process $\hat{\mathcal{P}}_{\rho'}$ one can give a much simpler proof of Theorem 1, that we describe in what follows. Denote by \mathcal{P} and $\mathcal{P}_{\rho'}$ the ν -randomization of $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_{\rho'}$, respectively. Due to the definition of stochastic domination given in Section 2, one can exhibit a coupling between \mathcal{P} and $\mathcal{P}_{\rho'}$ such that $\xi \subset \xi'$ almost surely, with (ξ, ξ') denoting the random sets with marginal distributions given by $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_{\rho'}$, respectively.*

Consider the graph $\mathcal{G}^\beta(\xi)$ introduced in Section 4. Due to Campbell identity (2.3), we can write

$$\mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^3 \right) = \frac{1}{\rho} \mathbf{E}_{\mathcal{P}} (F_\beta(\xi)), \quad F_\beta(\xi) := \int_{Q_1} \hat{\xi}(dx) \left(|C_0^\beta(S_x \xi)|^3 \right).$$

Since $F_\beta(\xi) \leq F_\beta(\xi')$ if $\xi \subset \xi'$ and due to the above coupling between \mathcal{P} and $\mathcal{P}_{\rho'}$, we can conclude that

$$\mathbf{E}_{\mathcal{P}} (F_\beta(\xi)) \leq \mathbf{E}_{\mathcal{P}_{\rho'}} (F_\beta(\xi)).$$

Due to Campbell identity (2.3), the r.h.s. in the above expression equals $\rho' \mathbf{E}_{\mathcal{P}_{0,\rho'}} \left(|C_0^\beta(\xi)|^3 \right)$ which, as proven in Section 4, is bounded from above uniformly in β . Hence we have that

$$\sup_{\beta > 0} \mathbf{E}_{\mathcal{P}_0} \left(|C_0^\beta(\xi)|^3 \right) \leq \frac{\rho'}{\rho} \sup_{\beta > 0} \mathbf{E}_{\mathcal{P}_{0,\rho'}} \left(|C_0^\beta(\xi)|^3 \right) < \infty. \quad (5.18)$$

At this point it is enough to apply Proposition 1: condition (3.5) is fulfilled due to (5.18), while all other conditions can be easily checked.

6. POINT PROCESSES WITH UNIFORM BOUNDS ON THE LOCAL DENSITY

Let us prove that the conditions of Theorem 1 are fulfilled by stationary point processes with uniform bounds:

Proposition 2. *Let $\hat{\mathcal{P}}$ be a stationary simple point process such that, for suitable positive constants K and N ,*

$$\hat{\xi}(\Lambda_K(x)) \leq N, \quad \forall x \in K\mathbb{Z}^d, \quad \hat{\mathcal{P}} \text{ a.s.} \quad (6.1)$$

where $\Lambda_K(x) = x + [-K/2, K/2]^d$.

Then, for each $q \in (0, 1]$, the q -thinning $\hat{\mathcal{P}}^{(q)}$ of $\hat{\mathcal{P}}$ satisfies condition (ii) in Theorem 1.

In order to prove the above statement, we use a standard result of stochastic domination (cf. Lemma 1.1 in [10]):

Lemma 4. *Let $p \in [0, 1]$ and let $\sigma = (\sigma_x : x \in \mathbb{Z}^d)$ be a random field s.t. $\sigma_x \in \{0, 1\}$ for all $x \in \mathbb{Z}^d$. Suppose that*

$$P(\sigma_x = 1 \mid \sigma = \zeta \text{ on } \mathbb{Z}^d \setminus \{x\}) \leq p, \quad (6.2)$$

for almost all $\zeta \in \{0,1\}^{\mathbb{Z}^d}$ and for all $x \in \mathbb{Z}^d$. Then σ is stochastically dominated by the Bernoulli site percolation with parameter p , i.e. one can define random fields (σ', ω) with

$$\sigma'_x \leq \omega_x \quad \forall x \in \mathbb{Z}^d, \quad \text{a.s.},$$

σ' having the same law of σ and ω being given by $\omega = (\omega_x : x \in \mathbb{Z}^d)$ where ω_x are i.i.d. random variables taking value 1 with probability p and value 0 with probability $1 - p$.

Proof of Proposition 2. If $q < 1$ set $p = q$, otherwise fix some $p \in (0, 1)$. Consider the random fields $\sigma = (\sigma_x : x \in \mathbb{Z}^d)$, $\tau = (\tau_x : x \in \mathbb{Z}^d)$ having value in $\{0, 1\}^{\mathbb{Z}^d}$ defined as

$$\sigma_x(\hat{\xi}) = \begin{cases} 1 & \text{if } \hat{\xi}(\Lambda_K(x)) \geq 1, \\ 0 & \text{otherwise;} \end{cases} \quad \tau_x(\hat{\xi}) = \begin{cases} 1 & \text{if } \hat{\xi}(\Lambda_K(x)) \geq N, \\ 0 & \text{otherwise.} \end{cases}$$

Then, given $\zeta \in \{0, 1\}^{\mathbb{Z}^d}$, we have

$$\begin{aligned} \hat{\mathcal{P}}^{(p)}(\sigma_x = 1 \mid \sigma = \zeta \text{ on } \mathbb{Z}^d \setminus \{x\}) &\leq 1 - (1 - p)^N =: p', \\ \hat{\mathcal{P}}_{\rho'}(\tau_x = 1 \mid \tau = \zeta \text{ on } \mathbb{Z}^d \setminus \{x\}) &= P(Z \geq N) =: \tilde{p}, \end{aligned}$$

where Z is a Poisson variable with mean $\rho' K^d$. Since $p' < 1$ we can choose ρ' large enough so that $p' \leq \tilde{p}$. Due to the previous lemma, we can conclude that the random field σ with $\hat{\xi}$ chosen with law $\hat{\mathcal{P}}^{(p)}$ is stochastically dominated by the site Bernoulli percolation with parameter \tilde{p} , which is stochastically dominated by the random field τ with $\hat{\xi}$ chosen with Poisson law $\hat{\mathcal{P}}_{\rho'}$. Due to the transitivity of stochastic domination and since

$$Y(x) \leq N\sigma_x, \quad N\tau_x \leq Y(x), \quad \forall x \in \mathbb{Z}^d,$$

we can conclude that the random field Y with $\hat{\xi}$ chosen with law $\hat{\mathcal{P}}^{(p)}$ is stochastically dominated by the random field Y with $\hat{\xi}$ chosen with law $\hat{\mathcal{P}}_{\rho'}$. \square

As corollary of Proposition 2, we obtain that Theorem 1 can be applied to crystals or diluted crystals. In order to be more precise, let us start with a crystal, i.e. (cf. [2]) a locally finite set $\Gamma \subset \mathbb{R}^d$ such that for a suitable basis v_1, v_2, \dots, v_d of \mathbb{R}^d , it holds

$$\Gamma - x = \Gamma \quad \forall x \in G := \{z_1 v_1 + z_2 v_2 + \dots + z_d v_d : z_i \in \mathbb{Z} \ \forall i\}. \quad (6.3)$$

Let Δ be the elementary cell

$$\Delta := \{t_1 v_1 + t_2 v_2 + \dots + t_d v_d : 0 \leq t_i < 1 \ \forall i\}.$$

Note that both the group G and the cell Δ depend on the basis v_1, v_2, \dots, v_d .

Let $\omega = (\omega_x : x \in \Gamma)$ be a site Bernoulli percolation on Γ with parameter $p \in (0, 1]$ and let V be a random vector independent of ω , chosen in the elementary cell Δ with uniform distribution. Then consider the simple point process

$$\hat{\zeta} := \sum_{x \in \Gamma} \omega_x \delta_{V+x},$$

obtained from the set Γ by a spatial randomization and a p -thinning, and call $\hat{\mathcal{P}}$ its law. The following holds:

Proposition 3. *The simple point process $\hat{\zeta}$ with law $\hat{\mathcal{P}}$ is stationary and does not depend on the specific basis v_1, v_2, \dots, v_d satisfying (6.3). Moreover, its Palm distribution $\hat{\mathcal{P}}_0$ is given by*

$$\hat{\mathcal{P}}_0 = \frac{1}{|\Delta \cap \Gamma|} \sum_{u \in \Delta \cap \Gamma} P_u \quad (6.4)$$

where P_u is the law of the simple point process

$$\hat{\zeta}_u := \delta_0 + \sum_{x \in \Gamma \setminus \{u\}} \omega_x \delta_{x-u}.$$

Since the Palm distribution $\hat{\mathcal{P}}_0$ depends only on $\hat{\mathcal{P}}$, the above result implies that $\hat{\mathcal{P}}_0$ does not depend on the specific basis v_1, v_2, \dots, v_d . If $p = 1$, the Palm distribution corresponds to choosing a point u of the crystal inside the elementary cell Δ with uniform probability and translating the crystal of $-u$. If $p < 1$, in addition to the previous step one erases points different from the origin independently with probability $1 - p$. The resulting simple point process is what we called p -diluted crystal obtained from Γ . Trivially, it has uniform bounds in the local density. Hence, due to Proposition 2, it fulfills the assumptions of Theorem 1.

Proof. Due to the translation invariance (6.3) one easily proves that $\hat{\mathcal{P}}$ is stationary. Let us prove that it does not depend on the specific basis v_1, v_2, \dots, v_d . Given a Borel subset $A \subset \mathbb{R}^d$ with finite positive Lebesgue measure, we define $\hat{\mathcal{P}}_A$ as the law of the simple point process $\sum_{x \in \Gamma} \omega_x \delta_{W+x}$, where ω is a site Bernoulli percolation on Γ with parameter p and W is a vector independent from ω and chosen in A with uniform probability. Note that $\hat{\mathcal{P}} = \hat{\mathcal{P}}_\Delta$. By means of (6.3) one can easily check that $\hat{\mathcal{P}}_\Delta = \hat{\mathcal{P}}_A$ if A is a union of sets of the form $\Delta + x$, $x \in G$. Now, consider the elementary cell Δ' associated to another basis v'_1, v'_2, \dots, v'_d satisfying (6.3). By the previous observation, $\hat{\mathcal{P}}_{\Delta'} = \hat{\mathcal{P}}_{N\Delta'}$ for each positive integer N . Define A_N as the union of all the sets of the form $\Delta + x$, $x \in G$, included in $N\Delta'$. Then $\hat{\mathcal{P}}_\Delta = \hat{\mathcal{P}}_{A_N}$. Finally observe that, given a bounded measurable function $f : \hat{\mathcal{N}} \rightarrow \mathbb{R}$, it holds

$$\left| \mathbf{E}_{\hat{\mathcal{P}}_{N\Delta'}}(f) - \mathbf{E}_{\hat{\mathcal{P}}_{A_N}}(f) \right| \leq C(\Delta, \Delta', f)/N.$$

Hence the same estimate holds with $\hat{\mathcal{P}}_{N\Delta'}$ and $\hat{\mathcal{P}}_{A_N}$ replaced by $\hat{\mathcal{P}}_{\Delta'}$ and $\hat{\mathcal{P}}_\Delta$, respectively. Taking $N \uparrow \infty$, we conclude that $\hat{\mathcal{P}}_{\Delta'} = \hat{\mathcal{P}}_\Delta$. Hence the law $\hat{\mathcal{P}}$ does not depend on the specific basis v_1, v_2, \dots, v_d satisfying (6.3).

We prove the characterization (6.4) of the Palm distribution $\hat{\mathcal{P}}_0$ by means of Campbell identity (2.2). Take a bounded measurable function $f : \hat{\mathcal{N}}_0 \rightarrow \mathbb{R}$ and denote by ρ the intensity of $\hat{\mathcal{P}}$. It is simple to check that

$$\rho = p|\Delta \cap \Gamma|/|\Delta|, \quad (6.5)$$

where $|\Delta \cap \Gamma|$ denotes the cardinality of $\Delta \cap \Gamma$ and $|\Delta|$ denotes the Lebesgue volume of Δ . Since (2.2) holds also with Q_K replaced by Δ , we get that

$$\mathbf{E}_{\hat{\mathcal{P}}_0}(f) = \frac{1}{\rho|\Delta|} \mathbf{E}_{\hat{\mathcal{P}}} \left(\int_{\Delta} \hat{\xi}(x) f(S_x \hat{\xi}) \right). \quad (6.6)$$

Let us define Γ_ω as the support of the measure $\sum_{x \in \Gamma} \omega_x \delta_x$. Denoting by \mathbf{E}_ω the expectation w.r.t. the site Bernoulli percolation ω we can write

$$\begin{aligned}
\mathbf{E}_{\hat{\mathcal{P}}} \left(\int_{\Delta} \hat{\xi}(x) f(S_x \hat{\xi}) \right) &= \mathbf{E}_\omega \left[\frac{1}{|\Delta|} \int_{\Delta} dy \left(\sum_{z \in \Delta \cap (\Gamma_\omega + y)} f(\Gamma_\omega + y - z) \right) \right] \\
&= \mathbf{E}_\omega \left[\frac{1}{|\Delta|} \int_{\Delta} dy \left(\sum_{u \in (\Delta - y) \cap \Gamma_\omega} f(\Gamma_\omega - u) \right) \right] \\
&= \mathbf{E}_\omega \left[\frac{1}{|\Delta|} \int_{\Delta} dy \left(\sum_{u \in (\Delta - y) \cap \Gamma} \omega_u f((\Gamma_\omega - u) \cup \{0\}) \right) \right] \quad (6.7) \\
&= p \mathbf{E}_\omega \left[\frac{1}{|\Delta|} \int_{\Delta} dy \left(\sum_{u \in (\Delta - y) \cap \Gamma} f((\Gamma_\omega - u) \cup \{0\}) \right) \right] \\
&= \frac{p}{|\Delta|} \int_{\Delta} dy \sum_{u \in (\Delta - y) \cap \Gamma} \mathbf{E}_\omega [f((\Gamma_\omega - u) \cup \{0\})] .
\end{aligned}$$

Due to (6.3), it is simple to check that the last summation does not depend on y . Hence from the above identities (6.5), (6.6) and (6.7) we get that

$$\begin{aligned}
\mathbf{E}_{\hat{\mathcal{P}}_0}(f) &= \frac{p}{\rho|\Delta|} \sum_{u \in \Delta \cap \Gamma} \mathbf{E}_\omega [f((\Gamma_\omega - u) \cup \{0\})] = \frac{p}{\rho|\Delta|} \sum_{u \in \Delta \cap \Gamma} \mathbf{E}_{P_u}(f) = \\
&\quad \frac{1}{|\Delta \cap \Gamma|} \sum_{u \in \Delta \cap \Gamma} \mathbf{E}_{P_u}(f), \quad (6.8)
\end{aligned}$$

thus concluding the proof of (6.4). □

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